2018

Existence of Global Solutions for Nonlinear Magnetohydrodynamics with Finite Larmor Radius Corrections

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Thesis submitted to the
Eberly College of Arts and Sciences
at West Virginia University
in partial fulfillment of the requirements for the degree of

Doctor of Philosophy
in
Mathematics

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Morgantown, West Virginia
2018

Keywords: Energy methods, Gronwall's inequality, Semigroups theory, A priori estimates

ABSTRACT

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We discuss the existence of global solutions to the MHD equations where the effects of finite Larmor radius corrections are taken into account. Unlike the usual MHD, the pressure is a tensor and it depends on not only the density but also the magnetic field. We show the existence of global solutions by the energy methods. Our techniques of proof are based on the existence of local solution by Semigroups theory and a priori estimates.

Acknowledgements

I praise God Almighty for granting me this opportunity and the ability to proceed successfully in my study. I would like to thank my mother for always keeping me in her prayers, my sisters and brothers for their endless support in all my endeavors.

First and foremost I thank my advisor, Professor Harumi Hattori with deep gratitude the considerable assistance we have received from. For his guidance and encouragement made it possible to complete the Ph.D. thesis and degree.

Secondly, I cannot express enough thanks to Dr. Diamond Harvey and the Department of Mathematics at West Virginia University for providing me with an excellent study environment and support during my study as a graduate student.

Thirdly, I would like to thank all my Professor committee members: Adrian Tudorascu, Charis Tsikkou, Casian Pantea and Tudor Stanescu.

Finally, I would like to thank all of the friends and colleagues that helped me reach this point particularly: Prof. John Goldwasser, for the support and encouragement he has given me to learn about combinatorics field, Dr. Tokunbo Fadahunsi from the Statistics department, Zola Donovan, Jesse Cook, my teacher Mrs. Mahboba Elmagrabi, and Nancy Smaltz, who offered her gracious assitance in any possible way.

Dedication

This work is dedicated to
my mother Halima, the soul of my father Abdelfrrag.

”If nature were not beautiful, it would not be worth studying it.
And life would not be worth living.”
Henry Poincare

Contents

Chapter 1

Introduction

Study of partial differential equations is of great importance in mathematics. Many applications in physics disscuced in mathematics are described by second order hyperbolic, parabolic equations, when the effects of dissipative mechanisms (such as viscosity, heat conduction, etc.) are taken into account. Nonlinear partial differential equations, and in particular the equations of fluid dynamics, are difficult to solve analytically and are not possible to solve exactly. Asymptotic behavior becomes important. We are interested in the global existence of solutions of one-dimensional magnetohydrodynamics (MHD) equations with finite Larmor raduis (FLR) corrections. We begin this chapter by providing a background in magnetohydromagnetics (MHD) and then extend our discussion to include finite Larmor corrections, gyro fluid (Landau fluid) and the connection between the physics problems and partial differential equations. After the basic concepts have been introduced, and later in this chapter we discuss related work concerning the existence of global solutions, and we introduce the preliminaries.

Magnetohydrodynamics (MHD) concerns the motion of conducting fluids, such as charged particles in an electromagnetic field. It also has an extensive range of applications in mathematics and physics. MHD has been the subject of many studies by physicists and mathematicians because of its environmental importance, rich phenomena, and mathematical challenges; see [1, 10, 16, 21]. It is treated as a subject in theoretical physics. When the magnetic field is strong, the charged particle undergoes gyro motion and it affects the fluid motion. Gyro motion comes into the fluid motion through the pressure tensor P. The radius of gyro motion is called the Larmor radius. The fluid description with the finite Larmor radius (FLR) effects is referred to as the gyro fluid, or the Landau fluid, and such effects are reflected in the pressure and the heat flux. Usually, in the MHD, the pressure (dependent on density) or (as a function of density) is isotropic and scalar. On the other hand, in the gyro fluid, the pressure is anisotropic and depends on both density and the magnetic field. As a result, the pressure is no longer a scalar. Instead, it is a tensor derived from the moment equations and it consists of a gyrotropic and gyroviscous tensor. The gyrotropic part consists of the pressure parallel and perpendicular to the magnetic field, and the gyroviscous tensor is derived from the moment equations for pressure by a Chapman-Enskog type expansion.

The main purpose of the present work is to show the existence of global solutions (in time) to the initial value problem for a hyperbolic-parabolic system. To establish the global solutions, we usually need local solutions and a priori estimates. We show the local solutions by constructing a sequence of approximation functions based on iteration. We apply semigroup theory for linearized equations for a small time. A semigroup can be used to solve a large class of evolution equations and the energy estimate (energy method) is based on Gronwall's inequality. A priori estimates refer to the fact that the estimates for the solution are derived before the solution is known to exist. To obtain a priori estimates, we make estimates on some terms using the Cauchy Schwarz inequality. Lastly, we show the existence of global solutions by extending the local solutions globally in time based on a priori estimates of solutions and we study the asymptotic behavior of solutions.

In recent years, the physical models have been studied from a mathematical point of view. As far as the derivation of gyro fluid is concerned, Chew, Goldberger, and Low [2] proposed the gyrotropic tensor. With no collision term, they deduced that the lowest order form of the plasma pressure can be expressed in terms of pressures parallel and perpendicular to the magnetic field. The gyroviscous tensor was first introduced by Thompson [26] and an improvement was made by Yajima [28], Khanna and Rajaram [15]. The general form of the gyroviscous tensor was derived by Hsu, Hazeltine, and Morrison [9]. Recently, Ramos [23] discussed the heat flux terms for the gyro fluid, and Passot and Sulem [20] discussed the closure relations for the gyro fluid. There are related papers that have been investigated by a number of authors that show the existence of solutions for MHD. Chen and Wang [1] showed the existence of global solutions to the piston problem. They considered a fundamental problem of the MHD fluid flow with the pressure $P = P(\rho, \theta)$, where ρ is a density and θ is a temperature. Additionally, they transformed the free boundary value problem into Lagrangian coordinates and established the existence and uniqueness of global solutions to the initial-boundary value problem with large initial data in H^1. Moreover, the existence of local solutions was based on the Banach theorem and the contractivity of the operator defined by the linearization of the problem on a small time interval. Hu and Wang [10] obtained the existence of global weak solutions for MHD equations in three dimensions by using an approximation scheme based on the Faedo-Galerkin method. Matsumura and Nishida [18] treated the initial value problem for equations of motion of viscous and heat conductive gases in three dimensions, and they obtained the existence of global solutions in H^3 with a method based on iteration and the energy method. Many applications of the energy method are discussed in Matsumura and Nishihara [19]. Slemrod [24] proved the existence of global solutions for nonlinear thermoelasticity with initial data sufficiently small and smooth in one dimension by applying the

contraction mapping theorem for the existence of local solutions.
In this regard, the existence of a global solution to partial differential equations of hyperbolic-parabolic type has received much attention in the last few decades by [4, 6, 11, 12, 22, 25]. Another method for obtaining the existence of solutions is by the difference approximation of solutions, see Hoff [7]. Some authors have succeeded to prove the global existence, uniqueness and asymptotic stability of smooth solutions for the physical systems, e-g. equations of compressible viscous fluids, see Hoff and Khodja [8] Kawashima and Okada [14].
Further, we are interested in the behavior of solutions as time tends to infinity. The asymptotic behavior of solutions of the Cauchy problem has been studied in several dimensions in [5, 8, 10, 13, 18, 24]. The H^s global existence and time-decay rate of strong solutions were obtained in whole space by Matsumura and Nishida [18]. Isentropic Navier-Stokes equations has been examined in [12]. The compressible Navier–Stokes system describing the one-dimensional motion of a viscous heat-conducting perfect polytropic gas in unbounded domains was studied in [17]. The long time behavior for general multi-dimensional hyperbolic-parabolic systems was studied in [13] with H^s time-decay rate of solutions.

We organize the rest of this thesis as follows. Chapter 2 presents a derivation of the equations of nonlinear Landau fluid in one dimension. Chapter 3 shows the existence of the local solutions in small time. In Chapter 4, we establish a priori estimates. In Chapter 5, the existence of global solutions is proved by extending the local solutions in time based on a priori estimates and we show the asymptotic behavior of solutions.

Notation and Preliminaries

Throughout this thesis, we introduce some notations for later use. Let Ω is a bounded domain in R^n.

Definition

For integer $1 < p < \infty$, $L^p(\Omega)$ the space of measurable functions f on Ω whose p-th powers are integrable with the norm

$$\|f\|_{L^p} = \left(\int_\Omega |f(x)|^p dx\right)^{\frac{1}{p}} < \infty,$$

when $p = 2$, we write $\|\cdot\|$ instead of $\|\cdot\|_{L^2}$.

For $p = \infty$, the space $L^\infty(\Omega)$ is the set of all measurable functions f from Ω to R which are essentially bounded, i.e. bounded up to a set of measure zero, with norm

$$\|f\|_\infty = \underset{0\le t\le T}{ess\ sup}\ |f|.$$

Or

$$\|f\|_{L^\infty} = \inf\{C \geq 0 : |f(x)| \leq C \quad \text{for alomost all} \;\; x \in \Omega\}$$

Definition

H^k denotes the space of functions all of whose derivatives of order $< k$ belong to $L^2(\Omega)$

$$H^k = \left\{f \in L^2 : D^{k-1} \in L^2\right\},$$

with norm

$$\|f\|_k = \left(f : \sum_{\alpha\le k} \int_\Omega |D^\alpha f|^p dx\right)^{\frac{1}{p}},$$

we will denote $\|\cdot\|_2$ is the H^2 norm.

Definition
$C^m(\Omega)$ is the space of the real valued functions on Ω that are m-times continuously differentiate.

$$C^m(\Omega) = \{f : C(\Omega) : D^\alpha f \in C(\Omega) \text{ for all } |\alpha| \le m\}.$$

C^∞ is the space of infinitely differentiable functions in Ω.

$$C^\infty(\Omega) = \{f : C(\Omega) : D^\alpha f \in C(\Omega) \text{ for all } \alpha\},$$

and

$$C_0^\infty(\Omega) = \{f : C^\infty(\Omega) : \text{ supp}(f) \text{ is compact}\}.$$

Cauchy's inequality with ε

$$ab \le \varepsilon a^2 + \frac{b^2}{4\varepsilon} \ (a, b > 0, \varepsilon > 0.).$$

Cauchy-Schwarz inequality

$$|x \cdot y| \le |x||y| \ (x, y \in R^n).$$

Definition
Afunction $f : R^n \to R$ is called convex provided

$$f(ax + (1-a)y) \le af(x) + (1-a)f(y),$$

for all $x, y \in R^n$ and each $0 \le a \le 1$.

Gronwall's inequality
Let $f(t)$ be a nonnegative, absolutaley continuous function on $[0,T]$ which satisfies for a.e t the differential inequality

$$f'(t) \le g(t)f(t) + k(t),$$

where $g(t)$ and $k(t)$ are nonnegative, summable functions on $[0,T]$ Then

$$f(t) \le \exp \int_0^t g(s)ds \left[f(0) + \int_0^t k(s)ds \right] \text{ for all } 0 \leqslant t \leqslant T.$$

Definition
Let f real valued function on the interval and $a = x_0 < x_1 < \cdots < x_n = b$ a partition of $[a,b]$.
Define

$$p(f,p) = \sum_{i=1}^{k} [f(x_i) - f(x_{i-1})]^+ ,$$

$$n(f,p) = \sum_{i=1}^{k} [f(x_i) - f(x_{i-1})]^- ,$$

$$v(f,p) = p + n = \sum_{i=1}^{n} |f(x_i) - f(x_{i-1})|.$$

Total Variation

$$TV(f) = \sup\{\ v(f,p) \mid p \text{ is a partation of } [a,b]\ \},$$

Bounded Variation
If $TV(f_{[a,b]}) < \infty$, we say f is of a bounded variation over $[a,b]$.

Chapter 2

Derivation of Landau Fluid Equations

2.1 Derivation

In this chapter, we look at the model for describing Landau fluid, which we derive from the following MHD equations in three dimensions; see for example [10]:

$$\rho_t + \nabla \cdot (u\rho) = 0,$$

$$(\rho u)_t + \nabla \cdot (\rho u \otimes u) + \nabla \cdot P = (\nabla \times b) \times b + \mu \Delta u + (\lambda + \mu)\nabla(\nabla \cdot u),$$

$$b_t - \nabla \times (u \times b) = -\nabla \times (\nu \nabla \times b), \ \nabla \cdot b = 0. \tag{2.1}$$

In the above system ρ is a density, $u = \langle u_1, u_2, u_3 \rangle$ is a velocity, ∇ is the gradient operator, $\otimes$ denotes the Kronecker tensor product, μ, λ are viscosity coefficients, $b = \langle b_1, b_2, b_3 \rangle$ is the magnetic field, and $\nu > 0$ is the magnetic diffusivity acting as a magnetic diffusion coefficient of the magnetic field. The viscosity coefficients satisfy

$$\mu > 0, \ 2\mu + 3\lambda > 0.$$

Usually, we refer to the first equation in (2.1) as the continuity equation, the second as the momentum equation, and the third as the magnetic field equation (called induction equation).

The magnetic field consists of the large ambient field oriented in the x-direction and small perturbation.
Equations for MHD are based on neglecting the displacement current in Maxwell's equations [1, 2, 9, 10]. It is well-known that the electromagnetic fields are governed by Maxwell's equations. Furthermore, it is convenient to write the electric field in terms of the magnetic field b and the velocity u,

$$\mathbf{E} = \nu \nabla \times b - u \times b.$$

By Hu and Wang [10], although the electric field $\mathbf{E}$ does not show up in (2.1), by the moving conductive flow in the magnetic field, this causes $\mathbf{E}$ to appear in the previous relation.
For any motion of the fluid there is a function $P(x,t)$ called the pressure. In the gyro fluid we assume that the pressure term P has the form

$$P = P^0 + P^1,$$

where P^0 and P^1 are called the gyrotropic tensor and the gyroviscosity tensor, respectively. For P^0, Chew, Goldberger and Low (CGL) [2] made no assumption that the pressure is a scalar, and they deduced that the lowest order form of the plasma pressure can be expressed in terms of pressures parallel and perpendicular to the magnetic field.

$$P^0 = p_\perp (I - \hat{b} \otimes \hat{b}) + p_{\shortparallel} \hat{b} \otimes \hat{b},$$

where $p_\perp$ is the perpendicular pressure, $p_{\shortparallel}$ is the parallel pressure, I is the identity matrix, and $\hat{b}$ is the unit vector in the direction of b.
Kulsrud [16] observed that the perpendicular pressure is propotioanl to

$$\frac{B}{\nu},$$

and parallel pressure is propotioanal to

$$\frac{1}{\nu^3 B^2},$$

i.e.,

$$p_\perp \propto \frac{B}{v}, \;\; p_{\shortparallel} \propto \frac{1}{v^3 B^2},$$

where $v = \frac{1}{\rho}$, v is specific volume, and $B = \sqrt{\sum_{i=1}^{3} b_i^2}$ is the magnitude of the magnetic field. Based on Kulsrud's observation, we use

$$p_\perp = \frac{B}{v}, \;\; p_{\shortparallel} = \frac{1}{v^3 B^2}.$$

These pressure laws can be interpretated in terms of the known behavior of individual charged particles in a strong magnetic field.
The gyroviscosity tensor P^1 (Finite Larmor radius correction (FLR)) is derived from ten moment equations, where the pressure tensor P satisfies

$$\frac{d}{dt}P + \nabla \cdot uP + (P \cdot \nabla)u + (P \cdot \nabla u)^T + \frac{e}{m}[\, b \times P - P \times b \,] = 0.$$

We write the above relation in the following way

$$\left[P \times \hat{b} + Tr\right] = \frac{1}{\Omega}\left[\frac{d}{dt}P + P\nabla \cdot u + P \cdot \nabla u + (P \cdot u)^T\right], \tag{2.2}$$

where $Tr = -\hat{b} \times P$ is the transpose of $P \times \hat{b}$ and

$$\frac{d}{dt} = \frac{\partial}{\partial t} + u \cdot \nabla,$$

denotes the convective derivative, $\Omega = \frac{\sigma B_0}{mc}$ is the gyrofrequency, where σ is the electron charge, m is the mass of particle, c is the speed of light, and B_0 is the constant ambient field assumed to be oriented in the x direction and approximately equal to B. In the gyro fluid, Ω is large. If we assume $P^0 = O(1)$ and $P^1 = O(\frac{1}{\Omega})$, then from (2.2) we obtain

$$\left[P^1 \times \hat{b} + Tr\right] \approx \frac{1}{\Omega}\left[\frac{d}{dt}P^0 + P^0\nabla \cdot u + P^0 \cdot \nabla u + (P^0 \cdot \nabla u)^T\right].$$

Solving the above equation for P^1, Hsu, Hazeltine and Morrison [9] obtained

$$P^1 = \frac{1}{4\Omega}\{\hat{b}\times\hat{S}\cdot(I+3\hat{b}\hat{b}) + [\hat{b}\times\hat{S}\cdot(I+3\hat{b}\hat{b})]^T\},$$

where

$$\hat{S} = \left(\frac{\partial}{\partial t} + u\cdot\nabla\right)P^0 + \Big[(P^0\cdot\nabla u) + Tr\Big].$$

So,

$$\begin{aligned}
P^1 = {} & \frac{1}{4\Omega}(p_{\shortparallel} - p_\perp)\{\hat{b}\times(\hat{b}\cdot\nabla u\,\hat{b})\cdot(I+3\,\hat{b}\,\hat{b}) - (I+3\,\hat{b}\,\hat{b})\cdot(\hat{b}\cdot\nabla u\,\hat{b})^T\times\hat{b}\} \\
& + \frac{1}{4\Omega}\{p_\perp\hat{b}\times\left(\nabla u + (\nabla u)^T\right)\cdot(I+3\,\hat{b}\,\hat{b}) + 4(p_{\shortparallel} - p_\perp)\,\hat{b}\times(\nabla u)^T\cdot\hat{b}\,\hat{b}\} \\
& + \frac{1}{4\Omega}\{-p_\perp(I+3\,\hat{b}\,\hat{b})\left(\nabla u + (\nabla u)^T\right)\times\hat{b} - 4(p_{\shortparallel} - p_\perp)\,\hat{b}\,\hat{b}\cdot\nabla u\times\hat{b}\},
\end{aligned}$$

Thus, we can obtain after a straightforward computation that, we have

$$P^1 = \begin{bmatrix}
\begin{gathered}\tfrac{(p_{\shortparallel}-p_\perp)}{\Omega}[4\hat{b_1}^2\hat{b_2}u_{3x} - 4\hat{b_1}^2\hat{b_3}u_{2x}] \\ +\tfrac{p_\perp}{4\Omega}[2(1+3\hat{b_1}^2)\hat{b_2}u_{3x} - 2(1+3\hat{b_1}^2)\hat{b_3}u_{2x}]\end{gathered} \\[2em]
\begin{gathered}\tfrac{(p_{\shortparallel}-p_\perp)}{\Omega}[2\hat{b_1}^2\hat{b_3}u_{1x} - 2\hat{b_1}^3u_{3x} - 2\hat{b_1}\hat{b_2}\hat{b_3}u_{2x} + 2\hat{b_1}\hat{b_2}^2u_{3x}] \\ +\tfrac{p_\perp}{4\Omega}[2(1+3\hat{b_1}^2)\hat{b_3}u_{1x} + 3\hat{b_1}\hat{b_3}u_{3x} \\ -(1+3\hat{b_1}^2)\hat{b_1}u_{3x} + 3\hat{b_1}\hat{b_2}u_{3x}]\end{gathered} \\[2em]
\begin{gathered}\tfrac{(p_{\shortparallel}-p_\perp)}{\Omega}[2\hat{b_1}\hat{b_2}\hat{b_3}u_{3x} - 2\hat{b_1}\hat{b_3}^2u_{2x} - 2\hat{b_1}^2\hat{b_2}u_{1x} + 2\hat{b_1}^3u_{2x}] \\ +\tfrac{p_\perp}{4\Omega}[(1+3\hat{b_1}^2)\hat{b_1}u_{2x} - 2(1+3\hat{b_1}^2)\hat{b_2}u_{1x} \\ +3\hat{b_1}\hat{b_2}^2u_{3x} + 3\hat{b_1}\hat{b_3}^2u_{2x}]\end{gathered}
\end{bmatrix}.$$

We discuss one dimensional motion, only the first column will be reveal after we take the divergence of P^1 the second and third column will be zero, and we reduced P^1 term as $P^1 = O(|b_1||\nabla u|) + O(|q||\nabla u|)$, where

$$O(|b_1|\nabla u) = \begin{bmatrix} 0 \\ \\ \frac{-2p_{\shortparallel}}{\Omega}\hat{b_1}^3 u_{3x} + \frac{2p_{\perp}}{\Omega}\hat{b_1}^3 u_{3x} + \frac{-p_{\perp}}{4\Omega}(1+3\hat{b_1}^2)\hat{b_1}u_{3x} \\ \\ \frac{2p_{\shortparallel}}{\Omega}\hat{b_1}^3 u_{2x} - \frac{2p_{\perp}}{\Omega}\hat{b_1}^3 u_{2x} + \frac{p_{\perp}}{4\Omega}(1+3\hat{b_1}^2)\hat{b_1}u_{2x} \end{bmatrix},$$

$$O(|q||\nabla u|) = \begin{bmatrix} \frac{(p_{\shortparallel}-p_{\perp})}{\Omega}[4\hat{b_1}^2\hat{b_2}u_{3x} - 4\hat{b_1}^2\hat{b_3}u_{2x}] \\ +\frac{p_{\perp}}{4\Omega}[2(1+3\hat{b_1}^2)\hat{b_2}u_{3x} - 2(1+3\hat{b_1}^2)\hat{b_3}u_{2x}] \\ \\ \frac{(p_{\shortparallel}-p_{\perp})}{\Omega}[2\hat{b_1}^2\hat{b_3}u_{1x} - 2\hat{b_1}\hat{b_2}\hat{b_3}u_{2x} + 2\hat{b_1}\hat{b_2}^2u_{3x}] \\ +\frac{p_{\perp}}{4\Omega}[2(1+3\hat{b_1}^2)\hat{b_3}u_{1x} + 3\hat{b_1}\hat{b_3}u_{3x} + 3\hat{b_1}\hat{b_2}u_{3x}] \\ \\ \frac{(p_{\shortparallel}-p_{\perp})}{\Omega}[2\hat{b_1}\hat{b_2}\hat{b_3}u_{3x} - 2\hat{b_1}\hat{b_3}^2u_{2x} - 2\hat{b_1}^2\hat{b_2}u_{1x}] \\ +\frac{p_{\perp}}{4\Omega}[(1+3\hat{b_1}^2)\hat{b_1}u_{2x} - 2(1+3\hat{b_1}^2)\hat{b_2}u_{1x} \\ +3\hat{b_1}\hat{b_2}^2u_{3x} + 3\hat{b_1}\hat{b_3}^2u_{2x}], \end{bmatrix},$$

and $q = \langle b_2, b_3 \rangle$.

The gyrovisous terms are dispersive as discussed in [21] and [28].
So, gyroviscosity becomes important when the Larmor radius of the ions becomes finite (but still small compared to the size of a fluid element), where $O(|b_1||\nabla u|)$ is the central part of the gyroviscous term (i.e. this term depends only on b_1) and $O(|q||\nabla u|)$ is a small correction part and a tensor.
In the one dimensional case, because the constraint $\nabla \cdot b = 0$ implies that b_1 is a constant and based on non-dimensionialzation, we may set $b_1 = 1$ without loss of generality. Equation (2.1) reduces to

$$\rho_t + (u_1\rho)_x = 0, \tag{2.3}$$

$$\rho\left(\begin{bmatrix}u_{1t}\\u_{2t}\\u_{3t}\end{bmatrix}+u_1\begin{bmatrix}u_{1x}\\u_{2x}\\u_{3x}\end{bmatrix}\right)+\left(\begin{bmatrix}p_\perp(1-\hat{b_1^2})+p_{\shortparallel}\hat{b_1^2}\\-p_\perp\hat{b_1}\hat{b_2}+p_{\shortparallel}\hat{b_1}\hat{b_2}\\-p_\perp\hat{b_1}\hat{b_3}+p_{\shortparallel}\hat{b_1}\hat{b_3}\end{bmatrix}+\begin{bmatrix}0\\\frac{1}{\Omega}(p_\perp-2p_{\shortparallel})\frac{u_{3x}}{v}\\-\frac{1}{\Omega}(p_\perp-2p_{\shortparallel})\frac{u_{2x}}{v}\end{bmatrix}\right)_x$$
$$+\begin{bmatrix}b_2b_{2y}+b_3b_{3y}\\-b_1b_{2y}\\-b_1b_{3y}\end{bmatrix}=\left(\begin{bmatrix}u_{1x}\\u_{2x}\\u_{3x}\end{bmatrix}\right)_x,$$
$$(2.4)$$

$$q_t+(u_1q-w)_x=(\nu q_x)_x,\ \ \nu=1, \tag{2.5}$$

where

$$w=\begin{bmatrix}u_2\\u_3\end{bmatrix}\quad q=\begin{bmatrix}b_2\\b_3\end{bmatrix},$$

Next, we show the derivation of our equations with Lagrangian variables $(y;T)$, we do change of variables $(x;t)\to(y;T)$ such that $y=y(x;T)$, where $y_x=\rho,\ y_t=-\rho\,u_1,\ y(0;0)=0$.
Here

$$X_t=X_T\,\frac{\partial T}{\partial t}+X_y\,\frac{\partial y}{\partial t}=X_T-\rho u_1X_y$$

$$X_x=X_T\,\frac{\partial T}{\partial x}+X_y\,\frac{\partial y}{\partial x}=\rho X_y$$

Where X is a function of u,v or q. By equation of (2.3). we set

$$y(x;t)=\int_0^x\rho(\xi,t)\,d\xi.$$

Then, we differentiate the above equation with respect to y.

$$1=\rho\,\frac{\partial x}{\partial y} \tag{2.6}$$

Then, we differentiate equation (2.6) with respect to t, we have

$$0 = \rho_t \, \frac{\partial x}{\partial y} + \rho \, \frac{\partial^2 x}{\partial y \partial t}, \tag{2.7}$$

where

$$\frac{\partial x}{\partial t} = u_1 \quad \text{and} \quad v = \frac{1}{\rho}.$$

So, equation (2.7) yields

$$0 = \left(\frac{1}{v}\right)_t \frac{\partial x}{\partial y} + \frac{1}{v}\frac{\partial}{\partial y}\left(\frac{\partial x}{\partial t}\right).$$

Take the derivative of the first term of the above equation, we have

$$v_t - u_{1y} = 0.$$

Therefore, we obtain the first equation of the system.

$$\begin{aligned} u_t + u_1 u_x &= u_T + u_y \cdot \frac{\partial y}{\partial t} + u_1 \frac{\partial u}{\partial y} \cdot \frac{\partial y}{\partial x} \\ &= u_T. \end{aligned} \tag{2.8}$$

We can derive the equations of magnetic field from the equation (2.5) and we follow the same calculations we did in (2.8).
We convert the equation (2.3), (2.4) and (2.5) to the Lagrangian coordinates so that there are no convective derivatives. This is because we take into account the fact that the fluid is moving and that the positions of the fluid particles change with time.

Going back from T to t, we assemble the equations in Lagrangian coordinates as follows:

$$v_t - u_{1y} = 0, \tag{2.9}$$

$$u_{1t} + (p_\perp(1-\hat{b_1^2}) + p_{\shortparallel}\hat{b_1^2})_y + \left(\frac{1}{2}(b_2^2+b_3^2)\right)_y = \left(\frac{u_{1y}}{v}\right)_y, \tag{2.10}$$

$$u_{2t}+(-p_{\perp}\hat{b_1}\hat{b_2}+p_{\shortparallel}\hat{b_1}\hat{b_2})_y+(-b_2)_y=\left(\frac{u_{2y}}{v}+\frac{1}{\Omega}(p_{\perp}-2p_{\shortparallel})\frac{u_{3y}}{v}\right)_y, \tag{2.11}$$

$$u_{3t}+(-p_{\perp}\hat{b_1}\hat{b_3}+p_{\shortparallel}\hat{b_1}\hat{b_3})_y+(-b_3)_y=\left(\frac{u_{3y}}{v}-\frac{1}{\Omega}(p_{\perp}-2p_{\shortparallel})\frac{u_{2y}}{v}\right)_y, \tag{2.12}$$

$$(vb_2)_t+(-u_2)_y=\left(\frac{b_{2y}}{v}\right)_y, \tag{2.13}$$

$$(vb_3)_t+(-u_3)_y=\left(\frac{b_{3y}}{v}\right)_y. \tag{2.14}$$

We discuss the Cauchy problem for the above equations with the initial data

$$(u,v-\bar{v},q)(0)=(u_0,v_0-\bar{v},q_0), \tag{2.15}$$

we consider $u=u(y,t), v=v(y,t)$ and $q=q(y,t)$ to be unknown functions of t and y, where $t>0$ and $y\in\mathbb{R}$, the initial data where v approaches a positive constant $\bar{v}>0$ as $y\to\pm\infty$, and the magnetic field b approaches (1, 0, 0) as $y\to\pm\infty$.

For this purpose we define the Banach space $X(J)$ as follows.

$$X(J)=\left\{\begin{array}{ll} & (u,v-\bar{v},q)\in C^0(J,H^2); \\ (u,v,q); & u_y,q_y\in L^2(J;H^2); \\ & v_y\in L^2(J;H^1); \\ & sup\|(u,v-\bar{v},q)(t)\|_2\le M,\ \inf v(y,t)\ge m,\ M,m>0, \end{array}\right.$$

where $J = [0, T]$ is the time interval, $\|\cdot\|_2$ is the H^2 norm, and we denote L^2 norm by $\|\cdot\|$.

2.2 Hyperbolicity of the first order terms

In this section, we show that the equations (2.9)-(2.15) are hyperbolic. The initial value problem of the system of equations (2.9)-(2.15) are equivalent to the first order system.

$$\begin{bmatrix} v \\ u_1 \\ u_2 \\ u_3 \\ b_2 \\ b_3 \end{bmatrix}_t = A \begin{bmatrix} v \\ u_1 \\ u_2 \\ u_3 \\ b_2 \\ b_3 \end{bmatrix}_y ,$$

where A is the operator defined by the matrix of the coefficients of the linear terms of the system (2.9)-(2.14).

$$A = \begin{bmatrix} 0 & -1 & 0 & 0 & 0 & 0 \\ \frac{-(B^2-1)}{v^2 B} - \frac{3}{v^4 B^4} & 0 & 0 & 0 & (\frac{(B^2+1)b_2}{vB^3} - \frac{4b_2}{v^3 B^6} + b_2) & (\frac{(B^2+1)b_3}{vB^3} - \frac{4b_3}{v^3 B^6} + b_3) \\ (\frac{b_2}{v^2 B} - \frac{3b_2}{v^4 B^4}) & 0 & 0 & 0 & (\frac{b_2^2}{vB^3} - \frac{4b_2^2}{v^3 B^6} - \frac{1}{vB} + \frac{1}{v^3 B^4} - 1) & (\frac{b_2 b_3}{vB^3} - \frac{4b_2 b_3}{v^3 B^6}) \\ (\frac{b_3}{v^2 B} - \frac{3b_3}{v^4 B^4}) & 0 & 0 & 0 & (\frac{b_2 b_3}{vB^3} - \frac{4b_2 b_3}{v^3 B^6}) & (\frac{b_3^2}{vB^3} - \frac{4b_3^2}{v^3 B^6} - \frac{1}{vB} + \frac{1}{v^3 B^4} - 1) \\ 0 & 0 & \frac{-1}{v} & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{-1}{v} & 0 & 0 \end{bmatrix}.$$

Since b_2 and b_3 are very small, we can assume that $b_2 = 0$ and $b_3 = 0$, with $B = 1$. Then, matrix A becomes

$$A = \begin{bmatrix} 0 & -1 & 0 & 0 & 0 & 0 \\ \frac{-3}{v^4} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{-1}{v} + \frac{1}{v^3} - 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{-1}{v} + \frac{1}{v^3} - 1 \\ 0 & 0 & \frac{-1}{v} & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{-1}{v} & 0 & 0 \end{bmatrix}$$

and

$$|A-\lambda I| = \begin{vmatrix} -\lambda & -1 & 0 & 0 & 0 & 0 \\ \frac{-3}{v^4} & -\lambda & 0 & 0 & 0 & 0 \\ 0 & 0 & -\lambda & 0 & \frac{-1}{v}+\frac{1}{v^3}-1 & 0 \\ 0 & 0 & 0 & -\lambda & 0 & \frac{-1}{v}+\frac{1}{v^3}-1 \\ 0 & 0 & \frac{-1}{v} & 0 & -\lambda & 0 \\ 0 & 0 & 0 & \frac{-1}{v} & 0 & -\lambda \end{vmatrix}$$

$$= \begin{vmatrix} -\lambda & -1 \\ \frac{-3}{v^4} & -\lambda \end{vmatrix} \begin{vmatrix} -\lambda & 0 & \frac{-1}{v}+\frac{1}{v^3}-1 & 0 \\ 0 & -\lambda & 0 & \frac{-1}{v}+\frac{1}{v^3}-1 \\ \frac{-1}{v} & 0 & -\lambda & 0 \\ 0 & \frac{-1}{v} & 0 & -\lambda \end{vmatrix}.$$

The eigenvalues λ are the solutions of the charactristic polynomial

$$|A-\lambda I| = \det(A-\lambda I) = 0,$$

$$\left(\lambda^2 - \frac{3}{v^4}\right)\left(\lambda^2 - \frac{1}{v}(\frac{1}{v} - \frac{1}{v^3} + 1)\right)^2 = 0.$$

It follows the eigenvalues of the system are

$$\lambda_{1,2} = \pm\left(\frac{3}{v^4}\right)^{\frac{1}{2}}$$

and

$$\lambda_{3,4,5,6} = \pm\left(\frac{1}{v^2} - \frac{1}{v^4} + \frac{1}{v}\right)^{\frac{1}{2}}$$

Moreover, the eigenvalues are real numbers. The system is hyperbolic provided $\left(\frac{1}{v^2} - \frac{1}{v^4} + \frac{1}{v}\right)$ is a positive number.
Therefore, the eigenvectors corresponding to the eigenvalues λ of A

such that $(A-\lambda I)X=0$ are

$$X_1=\begin{bmatrix}1\\-(\frac{3}{v^4})^{\frac{1}{2}}\\0\\0\\0\\0\end{bmatrix},\ X_2=\begin{bmatrix}1\\(\frac{3}{v^4})^{\frac{1}{2}}\\0\\0\\0\\0\end{bmatrix},$$

$$X_3=\begin{bmatrix}0\\0\\-(1-\frac{1}{v^2}+v)^{\frac{1}{2}}\\0\\1\\0\end{bmatrix},\ X_4=\begin{bmatrix}0\\0\\0\\(1-\frac{1}{v^2}+v)^{\frac{1}{2}}\\0\\1\end{bmatrix},$$

$$X_5=\begin{bmatrix}0\\0\\(1-\frac{1}{v^2}+v)^{\frac{1}{2}}\\0\\1\\0\end{bmatrix},\ X_6=\begin{bmatrix}0\\0\\0\\-(1-\frac{1}{v^2}+v)^{\frac{1}{2}}\\0\\1\end{bmatrix}.$$

Chapter 3

Existence of Local Solutions

In this chapter, we show the proof of the existence of local solutions by constructing a sequence of functions that converges to a function satisfying the Cauchy problem. This method gives a sequence of approximations to the solution (u,v,q) where the n-th approximation is obtained from one or more previous approximation(s). We linearize (2.9)-(2.14) and consider the following iteration.

$$v_t^{(n)} - u_{1y}^{(n-1)} = 0, \tag{3.1}$$

$$\begin{aligned} &u_{1t}^{(n)} + \left(p_{\perp}^{(n-1)}(1-(\hat{b_1}^{(n-1)})^2) + p_{\shortparallel}^{(n-1)}(\hat{b_1}^{(n-1)})^2\right)_y \\ &+ \left(\frac{1}{2}(b_2^{(n-1)})^2 + (b_3^{(n-1)})^2\right)_y = \left(\frac{u_{1y}^{(n)}}{v^{(n-1)}}\right)_y, \end{aligned} \tag{3.2}$$

$$\begin{aligned} &u_{2t}^{(n)} + \left(-p_{\perp}^{(n-1)}\hat{b_1}^{(n-1)}\hat{b_2}^{(n-1)} + p_{\shortparallel}^{(n-1)}\hat{b_1}^{(n-1)}\hat{b_2}^{(n-1)}\right)_y \\ &+ (-b_2^{(n-1)})_y = \left(\frac{u_{2y}^{(n)}}{v^{(n-1)}} + \frac{1}{\Omega}(p_{\perp}^{(n-1)} - 2p_{\shortparallel}^{(n-1)})\frac{u_{3y}^{(n)}}{v^{(n-1)}}\right)_y, \end{aligned} \tag{3.3}$$

$$
\begin{aligned}
&u_{3t}^{(n)} + \left(-p_{\perp}^{(n-1)}\hat{b_1}^{(n-1)}\hat{b_3}^{(n-1)} + p_{\shortparallel}^{(n-1)}\hat{b_1}^{(n-1)}\hat{b_3}^{(n-1)}\right)_y \\
&+(-b_3^{(n-1)})_y = \left(\frac{u_{3y}^{(n)}}{v^{(n-1)}} - \frac{1}{\Omega}(p_{\perp}^{(n-1)} - 2p_{\shortparallel}^{(n-1)})\frac{u_{2y}^{(n)}}{v^{(n-1)}}\right)_y,
\end{aligned} \tag{3.4}
$$

$$
(v^{(n-1)}b_2^{(n)})_t + (-u_2^{(n-1)})_y = \left(\frac{1}{v^{(n-1)}}b_{2y}^{(n)}\right)_y, \tag{3.5}
$$

$$
(v^{(n-1)}b_3^{(n)})_t + (-u_3^{(n-1)})_y = \left(\frac{1}{v^{(n-1)}}b_{3y}^{(n)}\right)_y. \tag{3.6}
$$

We prove the existence of solutions for the linearized equations by semigroup theory. Since it is cumbersome to carry the superscripts $(n),(n-1)$, we drop them for simplicity. However, we may place the superscripts if it is necessary. Also for $(n-1)$ and the initial data circumvent the problem with the lack of regularity, we approximate $v^{(n-1)}$, u_0, $q^{(n-1)}$ and $g^{(n-1)}$ by smooth functions $\{v_l\}_{l=1}^{\infty}, \{u_{0l}\}_{l=1}^{\infty}$, $\{q_l\}_{l=1}^{\infty}$ and $\{g_l\}_{l=1}^{\infty}$ satisfying (3.7), (3.8).

$$
\begin{cases}
u_{0l} \in C_0^{\infty} \text{ and } u_{0l} \to u_0 \ strongly \ in \ H^2, \\
v_l - \bar{v} \in C^1(J; C_0^{\infty}), \\
v_l - \bar{v} \to v^{(n-1)} - \bar{v} \ \ strongly \ in \ C^0(J;H^2) \cap C^1(J;H^1), \\
q_l \in C^1(J;C_0^{\infty}), \ \ q_l \to q^{(n-1)} \ \ strongly \ in \ H^2, \\
g_l \in C^0(J;C_0^{\infty}), \ \ g_l \to g^{(n-1)} \ strongly \ in \ C^0(J;L^2),
\end{cases} \tag{3.7}
$$

$$
\begin{cases} \sup\|(v_l - \bar{v}, u_l, q_l)(t)\|_2 \leq M, \;\; \inf\; (v_l)(y,t) \geq m, \\ \\ \|u_{0l}\|_2 \leq \|u_0\|_2, \;\; \sup\|g_l(t)\| \leq \sup\|g^{(n-1)}(t)\|. \end{cases} \tag{3.8}
$$

For example, $g^{(n-1)}(t)$ of equation (3.2) is

$$
g^{(n-1)}(t) = -\left[p_{\perp}^{(n-1)}(1-\hat{b}_1^{2(n-1)}) + p_{\shortparallel}^{(n-1)}\hat{b}_1^{2(n-1)}\right]_y
$$

$$
-\left[\frac{1}{2}\left(b_2^{2(n-1)} + b_3^{2(n-1)}\right)\right]_y .
$$

Theorem 1. Existence and Energy Estimates for Linearized equations For each $m > 0$ and $M > 0$ there exist $T > 0$ and C depending on m and M, such that $U \in C^0(J;H^2)$ and that the initial value problem with the initial data $(u_0, v_0 - \bar{v}, q_0)$ for linearized equations (3.1)-(3.6) have unique solution satisfying the following energy estimiates.

$$
\begin{cases} \|(u,q)(t)\|_2^2 \leq \exp^{C(m,M)t}\left(\|(u,q)(0)\|_2^2 + C(m,M)\int_0^t \|g\|_1^2 d\tau\right), \\ \\ \int_0^t \|(u_y,q_y)(\tau)\|_2^2 d\tau \leq \exp^{C(m,M)t}\left(\|(u,q)(0)\|_2^2 + C(m,M)\int_0^t \|g\|_1^2 d\tau\right), \\ \\ \|v(t)\|_2^2 \leq \exp^{Ct}\left(\|v(0)\|_2^2 + \int_0^t \|h\|_1^2 d\tau\right), \end{cases} \tag{3.9}
$$

where g and h are known functions belonging to $X(J)$.

Proof. We use semigroup theory on a linear system of equations (3.1)-(3.6) and apply Gronwall's inequality to obtain the energy estimates (3.9).
Equations (3.2)-(3.6), except (3.1), can be expressed as system of equations that have the following form

$$z_t + A(z_{(l)})z = g_{(l)},$$

where z represents u or q in our calculations and $A(z_{(l)})z$ is an elliptic operator. For $z = (u_1, u_2, u_3)$, $A(z_{(l)})z$ maybe given by

$$A(z_{(l)})z = \left[\begin{bmatrix} \frac{1}{v_l} & 0 & 0 \\ 0 & \frac{1}{v_l} & \frac{1}{\Omega\, v}(p_{l\perp} - 2p_{l\shortparallel}) \\ 0 & \frac{-1}{\Omega\, v_l}(p_{l\perp} - 2p_{l\shortparallel}) & \frac{1}{v_l} \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix}_y \right]_y .$$

We treat u_1, u_2 and u_3 as the main part for the second order terms of $A(z_{(l)})z$ and the other terms of P^1 are small and does not effect the ellipticity of the operator. Since there is a constant $\gamma > 0$ such that $A(z_{(l)})$ satisfies

$$\sum_{i=1}^{3}\sum_{j=1}^{3} A_{ij}\ \varphi_i\varphi_j \geq \gamma\,|\varphi|^2, \text{where } \varphi \in \mathbb{R}, \tag{3.10}$$

by semigroup theory the solution exists for U.
For the energy estimates, we show the estimates for u_1 and v, since the analysis of u_2, u_3 and q can be carried out in a similar way. For u_1 equation (3.2) after the modifications can be expressed as

$$u_{1t} - \left(\frac{u_{1y}}{v_l}\right)_y = g_{(l)}, \tag{3.11}$$

where v_l, g_l are the sequences defined in (3.7) and (3.8).
First, we shall obtain the energy estimates for linear equations. We first show the estimate (3.9) under the assumptions that U_l satisfies (3.7) and (3.8) and that U is a solution of (3.11) and satisfying (3.7).

We multiply equation (3.11) by u_1 and integrate with respect to y by using integration by parts on the left hand-side. We obtain

$$\int_R \frac{1}{2}(u_1^2)_t dy - \int_R u_1 \left(\frac{u_{1y}}{v_l}\right)_y dy = \int_R u_1 g_l dy.$$

We estimate the term on right-hand side of the above equation by the Cauchy inequality. This gives

$$\frac{d}{dt}\left(\int_R \frac{1}{2}u_1^2 dy\right) + \int_R \frac{u_{1y}^2}{v_l} dy \leq \int_R |u_1||g_l| dy,$$

where $sup\|v_l - \bar{v}\|_1 \leq M$. We then integrate the above inequality with respect to t. By using the Cauchy Schwarz inequality on the term on the right-hand side, we obtain

$$\|u_1(t)\|^2 + \theta(M)\int_0^t \|u_{1y}\|^2 d\tau \leq \|u_1(0)\|^2 + \frac{1}{2}\int_0^t \left(\|u_1\|^2 + \|g_l\|^2\right) d\tau, \tag{3.12}$$

where $\theta(M) > 0$.
Next, we show the estimate for u_{1y}. We multiply equation (3.11) by $-u_{1yy}$ and integrate with respect to y; we obtain

$$-\int_R u_{1yy}u_{1t} dy + \int_R u_{1yy}\left(\frac{u_{1y}}{v_l}\right)_y dy = -\int_R u_{1yy}g_l dy.$$

Taking the first derivative with respect to y of the second term on the left-hand side and applying the Cauchy inequality of the term on the right-hand side of the above equation, yields

$$\int_R \frac{1}{2}(u_{1y}^2)_t dy + \int_R u_{1yy}\left(\frac{u_{1yy}}{v_l} - \frac{u_{1y}v_{ly}}{v_l^2}\right) dy = \int_R |u_{1yy}g_l| dy,$$

Equivalently, we have

$$\frac{d}{dt}\left(\int_R \frac{1}{2}u_{1y}^2 dy\right) + \int_R \frac{u_{1yy}^2}{v_l} dy \leq \int_R \frac{u_{1y}u_{1yy}v_{ly}}{v_l^2} dy + \int_R |u_{1yy}||g_l| dy. \tag{3.13}$$

Next, by using the Cauchy Schwarz inequality of the first term on the right-hand side of the above equation, we have

$$
\begin{aligned}
|\int_R \frac{u_{1y}u_{1yy}v_{ly}}{v_l^2}dy| &\le \varepsilon \int_R u_{1yy}^2 dy + \frac{C}{\varepsilon}\int_R \frac{|u_{1y}|^2|v_{ly}|^2}{v_l^4} \\
&\le \varepsilon \int_R u_{1yy}^2 dy + \frac{C}{\varepsilon m^4}|u_{1y}|_\infty^2 \int_R v_{ly}^2 dy \\
&\le \varepsilon \int_R u_{1yy}^2 dy + \frac{C(m,M)}{\varepsilon^2}\int_R \|u_{1y}\|\|u_{1yy}\| dy \\
&\le \varepsilon \int_R u_{1yy}^2 dy + \frac{C(m,M)}{\varepsilon^2}\int_R |u_{1y}|^2 dy.
\end{aligned}
$$

By integrating equation (3.13) with respect to t, after some straightforward steps we obtain

$$
\|u_{1y}(t)\|^2 + \theta(M)\|u_{1yy}\|^2 \le \|u_{1y}(0)\|^2 + \frac{C(m,M)}{\varepsilon^2}\left(\|u_{1y}\|^2 + \|g_l\|^2\right). \tag{3.14}
$$

Next, we show the estimate for u_{1yy}. We differentiate equation (3.11) in y and multiply by $-u_{1yyy}$, then we integrate with respect to y, and obtain

$$
-\int_R u_{1yyy}u_{1ty}dy + \int_R u_{1yyy}\left(\frac{u_{1y}}{v_l}\right)_{yy} dy = -\int_R u_{1yyy}g_{ly}dy.
$$

By taking the second derivative with respect to y of the second term on the left-hand side, we have

$$
\int_R u_{1tyy}u_{1yy}dy + \int_R u_{1yyy}\left(\frac{u_{1yyy}}{v_l} - \frac{2u_{1yy}v_{ly}}{v_l^2} - \frac{u_{1y}v_{lyy}}{v_l^2} - \frac{2u_{1y}v_{ly}^2}{v_l^3}\right) dy
$$
$$
= -\int_R u_{1yyy}g_{ly}dy.
$$

Then, we integrate the above equation with respect to t, and obtain

$$\begin{aligned}
&\int_0^t \int_R \left(\frac{1}{2}u_{1yy}\right)^2_t dyd\tau + \int_0^t \int_R \frac{u_{1yyy}^2}{v_l} dyd\tau \\
&\leq \int_0^t \int_R \frac{2u_{1yy}v_y u_{1yyy}}{v_l^2} dyd\tau + \int_0^t \int_R \frac{2u_{1y}v_{ly}^2 u_{1yyy}}{v_l^3} dyd\tau \qquad (3.15) \\
&+ \int_0^t \int_R \frac{u_{1y}v_{lyy}u_{1yyy}}{v_l^2} dyd\tau + \int_0^t \int_R \left(\frac{1}{2}|u_{1yyy}|^2 + \frac{1}{2}|g_{ly}|\right) dyd\tau.
\end{aligned}$$

We estimate the terms on the right-hand side from the above equation by using the Cauchy Schwarz inequality. We estimate the third term as follows.

$$\begin{aligned}
&\int_0^t \int_R \frac{u_{1y}v_{lyy}u_{1yyy}}{v_l^2} dyd\tau \\
&\leq \varepsilon \int_0^t \int_R u_{1yyy}^2 dyd\tau + \frac{C(m,M)}{\varepsilon} \int_0^t \|u_{1y}\|^2 \|u_{1yy}\|^2 d\tau.
\end{aligned}$$

Equation (3.15) becomes

$$\begin{aligned}
&\|u_{1yy}(t)\|^2 + \theta(M) \int_0^t \|u_{1yyy}\|^2 d\tau \\
&\leq \|u_{1yy}(0)\|^2 + \frac{C(m,M)}{\varepsilon} \int_0^t \left(\|u_{1yy}\|^2 + \|g_{ly}\|^2\right) d\tau. \qquad (3.16)
\end{aligned}$$

We combine equation (3.12), (3.14) and (3.16). After some straightforward steps we obtain

$$\|u_1(t)\|^2+\|u_{1y}(t)\|^2+\|u_{1yy}(t)\|^2$$

$$+\theta(M)\int_0^t\|u_{1y}\|^2d\tau+\theta(M)\int_0^t\|u_{1yy}\|^2d\tau+\theta(M)\int_0^t\|u_{1yyy}\|^2d\tau$$

$$\leq\|u_1(0)\|^2+\|u_{1y}(0)\|^2+\|u_{1yy}(0)\|^2$$

$$+\frac{C(m,M)}{\varepsilon}\int_0^t\left[\|u_1\|^2+\|u_{1y}\|^2+\|u_{1yy}\|^2+\|g_l\|^2+\|g_{ly}\|^2\right]d\tau.$$

We have similar way to estimate u_2 and u_3.Then, Gronwall's inequality implies that

$$\|u(t)\|_2^2\leq\exp^{C(m,M)t}\left[\|u(0)\|_2^2+C(m,M)\int_0^t\left(\|g_l\|_1^2\right)\right]d\tau. \quad (3.17)$$

The similar estimates can be obtained for q.
Next, we establish the energy estimates for v by multiplying equation (3.1) by v and integrate with respect to y and t. We obtain

$$\|v(t)\|^2\leq\|v(0)\|^2+\frac{1}{2}\int_0^t\left(\|v\|^2+\|h\|^2\right)d\tau. \quad (3.18)$$

We multiply (3.1) by $-v_{yy}$ and integrate with respect to y and t, and obtain

$$\|v_y(t)\|^2\leq\|v_y(0)\|^2+\frac{1}{2}\int_0^t\left(\|v_{yy}\|^2+\|h\|^2\right)d\tau. \quad (3.19)$$

We differentiate equation (3.1) in y and multiply by $-v_{yyy}$, then we integrate with respect to y and t, and obtain

$$\|v_{yy}(t)\|^2\leq\|v_{yy}(0)\|^2+\frac{1}{2}\int_0^t\left(\|v_{yy}\|^2+\|h_{yy}\|^2\right)d\tau. \quad (3.20)$$

We combine equations (3.18), (3.19) and (3.20). We have

$$\|v(t)\|_2^2 \leq \|v(0)\|_2^2 + \frac{1}{2}\int_0^t \left(\|h\|^2 + \|h_{yy}\|_1^2\right) d\tau.$$

By applying Gronwall's inequality on the above equation, we obtain

$$\|v(t)\|_2^2 \leq \exp^{Ct}\left(\|v(0)\|_2^2 + \int_0^t \|h\|_1^2 d\tau\right).$$

Combining the above estimates, we arrive at the energy estimates. (3.9) With the energy estimates (3.9). We see that the solution obtained by semigroup is in H^2. It is possible to show that for small T the sequence $\{U^l\}$ is a Cauchy sequence in $C^0(J;H)$. Therefore, the limit exists and we denote it as $U^{(n)}$. ■

The uniqueness of the solution follows from the energy estimates.
Now, we state a theorem for the existence of local solutions for nonlinear equations (2.9)-(2.14).

Theorem 2. Existence of Local Solutions for Nonlinear Equations For all $m > 0$ and $M > 0$, there exists $T(m,M)$ such that if $\|u_0, v_0 - \bar{v}, q_0\|_2 \leq M$, $\inf\ v(y,0) \geq m$, then the initial value problem for equations (2.9)-(2.14) has a local solution in X(J).

Proof. The proof is based on the approximation problem in theorem 1. It is not difficult to show that $\{U^{(n)}\}$ with the initial data

$$(u^{(n)}, v^{(n)}, q^{(n)})(y,0) = (u_0, v_0, q_0)$$

is the Cauchy sequence in $C^0(J;H^2)$, which converges to the solution of the Cauchy problem (2.9)-(2.15). Then the limit U is the local solution for the Cauchy problem (2.9)-(2.15).
We can obtain the uniform bound on the sequence $\{U^{(n)}\}$ from the energy estimates (3.9). ■

As a consequence, the one-dimensional equations (2.9)-(2.14) admit a unique local solution $C^0(J,H^2)$ provided that the initial data are in H^2.

Chapter 4

A Priori Estimates

The main concern of this chapter is to provide some estimates for the solutions for any fixed $t > 0$. To prove the existence of global solutions we need a priori estimates where we estimate u, v, and q, and the first and second derivatives of them. We define the energy and dissipation terms as $E(t)$ and $F(t)$ respectively.

$$E(t) = \frac{1}{2}\left(\|u\|_2^2 + \|q\|_2^2 + \|(v-\bar{v})\|_2^2\right),$$

and

$$F(t) = \frac{1}{2}\int_0^t \left(\|u_y\|_2^2 + \|q_y\|_2^2 + \|(v-\bar{v})_y\|_1^2\right) d\tau,$$

First, we multiply equations (2.10), (2.11), and (2.12) by u_1, u_2, u_3, respectively, and we perform integration by parts with respect to y. Then, by adding the resulting integrals, we obtain

$$
\begin{aligned}
&\int_{-\infty}^{\infty}\sum_{i=1}^{3}\frac{1}{2}(u_i^2)_t dy-\int_{-\infty}^{\infty}u_{1y}\left(p_\perp(1-\hat{b_1^2})+p_{\shortparallel}\hat{b_1^2}-\bar{p}_{\shortparallel}\right)dy\\
&-\int_{-\infty}^{\infty}u_{2y}\left(-p_\perp\hat{b_1}\hat{b_2}+p_{\shortparallel}\hat{b_1}\hat{b_2}\right)dy\\
&-\int_{-\infty}^{\infty}u_{3y}\left(-p_\perp\hat{b_1}\hat{b_3}+p_{\shortparallel}\hat{b_1}\hat{b_3}\right)dy\\
&+\int_{-\infty}^{\infty}u_{2y}b_2dy+\int_{-\infty}^{\infty}u_{3y}b_3dy \qquad (4.1)\\
&=\int_{-\infty}^{\infty}\frac{1}{2}u_{1y}(b_2^2+b_3^2)dy-\int_{-\infty}^{\infty}u_{1y}\left(\frac{u_{1y}}{v}\right)dy\\
&-\int_{-\infty}^{\infty}u_{2y}\left(\frac{u_{2y}}{v}+\frac{1}{\Omega}(p_\perp-2p_{\shortparallel})\frac{u_{3y}}{v}\right)dy\\
&-\int_{-\infty}^{\infty}u_{3y}\left(\frac{u_{3y}}{v}-\frac{1}{\Omega}(p_\perp-2p_{\shortparallel})\frac{u_{2y}}{v}\right)dy,
\end{aligned}
$$

where $\bar{p}_{\shortparallel}=p_{\shortparallel}(\bar{v},\bar{B})$ is a constant of integration, $\bar{B}=1$. Note that the two terms containing Ω are canceled and they cause no problem in a priori estimates. To carry out the estimates, we use equations (2.9), (2.13), and (2.14) for u_{1y},u_{2y},u_{3y}, and the relations

$$\hat{b_1}=\frac{1}{B},\ \hat{b_2}=\frac{b_2}{B}\ \text{ and }\ \hat{b_3}=\frac{b_3}{B}.$$

Equation (4.1), becomes

$$\int_{-\infty}^{\infty}\sum_{i=1}^{3}\frac{1}{2}(u_i^2)_t dy+\int_{-\infty}^{\infty}u_{2y}b_2 dy+\int_{-\infty}^{\infty}u_{3y}b_3 dy$$

$$-\int_{-\infty}^{\infty}v_t\left(p_\perp(1-\frac{1}{B^2})+p_{\shortparallel}\frac{1}{B^2}-\bar{p}_{\shortparallel}\right)dy$$

$$+\int_{-\infty}^{\infty}\left[(vb_2)_t-(\frac{1}{v}b_{2y})_y\right]\left(-p_\perp\frac{b_2}{B^2}+p_{\shortparallel}\frac{b_2}{B^2}\right)dy$$

$$+\int_{-\infty}^{\infty}\left[(vb_3)_t-(\frac{1}{v}b_{3y})_y\right]\left(-p_\perp\frac{b_3}{B^2}+p_{\shortparallel}\frac{b_3}{B^2}\right)dy$$

$$=\int_{-\infty}^{\infty}\frac{1}{2}u_{1y}(b_2^2+b_3^2)dy-\int_{-\infty}^{\infty}\frac{u_{1y}^2}{v}dy$$

$$-\int_{-\infty}^{\infty}\frac{u_{2y}^2}{v}dy-\int_{-\infty}^{\infty}\frac{u_{3y}^2}{v}dy,$$

we combine the fourh, the fifth and the sisth terms (pressure terms) on the left-hand side of the above equation. Then, the sum of the three terms leads to

$$=-\int_{-\infty}^{\infty}v_t\frac{1}{v^3B^2}dy+\int_{-\infty}^{\infty}v_t\frac{1}{\bar{v}^3\bar{B}^2}dy-\int_{-\infty}^{\infty}\frac{1}{2}v(-p_\perp+p_{\shortparallel})\frac{(B^2)_t}{B^2}dy$$

$$+\int_{-\infty}^{\infty}(\frac{1}{v}b_{2y})_y(-p_\perp+p_{\shortparallel})\frac{b_2}{B^2}dy+\int_{-\infty}^{\infty}(\frac{1}{v}b_{3y})_y(-p_\perp+p_{\shortparallel})\frac{b_3}{B^2}dy,$$

yields,

$$\int_{-\infty}^{\infty}\left(\frac{1}{2v^2B^2}\right)_t dy+\int_{-\infty}^{\infty}v_t\frac{1}{\bar{v}^3\bar{B}^2}dy+\int_{-\infty}^{\infty}B_t dy$$

$$+\int_{-\infty}^{\infty}\left(\frac{1}{v}b_{2y}\right)_y\left(-\frac{B}{v}+\frac{1}{v^3B^2}\right)\frac{b_2}{B^2}dy+\int_{-\infty}^{\infty}\left(\frac{1}{v}b_{3y}\right)_y\left(-\frac{B}{v}+\frac{1}{v^3B^2}\right)\frac{b_3}{B^2}dy.$$

we now integrate with respect to t and expand the term $\frac{1}{2v^2B^2}$ by the Taylor series. Then, linear terms for v cancel, and we have

$$\begin{aligned}
&\int_{-\infty}^{\infty}\frac{1}{2}\sum_1^3 u_i^2(y,t)dy+\int_{-\infty}^{\infty}(1-\frac{1}{\bar{v}^2\bar{B}^3})(B-\bar{B})dy\\
&+\int_{-\infty}^{\infty}\frac{1}{2!}\left(\frac{3}{\bar{v}^4\bar{B}^2}(v-\bar{v})^2+\frac{2}{\bar{v}^3\bar{B}^3}(v-\bar{v})(B-\bar{B})+\frac{3}{\bar{v}^2\bar{B}^4}(B-\bar{B})^2\right)dy\\
&+\int_0^t\int_{-\infty}^{\infty}u_{2y}b_2dyd\tau+\int_0^t\int_{-\infty}^{\infty}u_{3y}b_3dyd\tau+\int_0^t\int_{-\infty}^{\infty}\frac{u_{1y}^2}{v}dyd\tau\\
&+\int_0^t\int_{-\infty}^{\infty}\frac{u_{2y}^2}{v}dyd\tau+\int_0^t\int_{-\infty}^{\infty}\frac{u_{3y}^2}{v}dyd\tau-\int_0^t\int_{-\infty}^{\infty}\frac{1}{2}u_{1y}(b_2^2+b_3^2)dyd\tau\\
&\le\int_{-\infty}^{\infty}\frac{1}{2}\sum_1^3 u_i^2(y,0)dy+\int_0^t\int_{-\infty}^{\infty}\left(\frac{1}{v}b_{2y}\right)_y\left(-\frac{B}{v}+\frac{1}{v^3B^2}\right)\frac{b_2}{B^2}dyd\tau\\
&+\int_0^t\int_{-\infty}^{\infty}\left(\frac{1}{v}b_{3y}\right)_y\left(-\frac{B}{v}+\frac{1}{v^3B^2}\right)\frac{b_3}{B^2}dyd\tau,
\end{aligned}\tag{4.2}$$

We note that for the third integral on the left-hand side of (4.2) the integrand is estimated from below in the following way.

$$
\begin{aligned}
&\frac{2}{\bar{v}^4\bar{B}^2}(v-\bar{v})^2+\frac{2}{\bar{v}^2\bar{B}^4}(B-\bar{B})^2\\
&\leq \frac{3}{\bar{v}^4\bar{B}^2}(v-\bar{v})^2+\frac{2}{\bar{v}^3\bar{B}^3}(v-\bar{v})(B-\bar{B})+\frac{3}{\bar{v}^2\bar{B}^4}(B-\bar{B})^2.
\end{aligned}
$$

Next, we estimate $q=\langle b_2,b_3\rangle$. For this purpose, we multiply equations (2.13) and (2.14) by b_2,b_3, respectively. Then we combine them and perform the integration with respect to $y$and t.

$$
\begin{aligned}
&\int_{-\infty}^{\infty}\frac{1}{2}vb_2^2(y,t)dy+\int_{-\infty}^{\infty}\frac{1}{2}vb_3^2(y,t)dy+\int_0^t\int_{-\infty}^{\infty}\frac{1}{v}b_{2y}^2dyd\tau\\
&+\int_0^t\int_{-\infty}^{\infty}\frac{1}{v}b_{3y}^2dyd\tau+\int_0^t\int_{-\infty}^{\infty}\frac{1}{2}u_{1y}(b_2^2+b_3^2)dyd\tau\\
&-\int_0^t\int_{-\infty}^{\infty}u_{2y}b_2dyd\tau-\int_0^t\int_{-\infty}^{\infty}u_{3y}b_3dyd\tau\\
&\leq\int_{-\infty}^{\infty}\frac{1}{2}vb_2^2(y,0)dy+\int_{-\infty}^{\infty}\frac{1}{2}vb_3^2(y,0)dy.
\end{aligned}
\tag{4.3}
$$

Next, to get the estimate for v_y, we multiply equation (2.10) by $\frac{v_y}{v}$, and integrate with respect to y, and we obtain

$$\int_{-\infty}^{\infty} \frac{v_y}{v} u_{1t}\, dy + \int_{-\infty}^{\infty} \frac{v_y}{v} \left(p_\perp (1 - \hat{b_1^2}) + p_{\shortparallel} \hat{b_1^2} \right)_y dy$$
$$+ \int_{-\infty}^{\infty} \frac{v_y}{v} \left(\frac{1}{2}(b_2^2 + b_3^2) \right)_y dy = \int_{-\infty}^{\infty} \frac{v_y}{v} \left(\frac{u_{1y}}{v} \right)_y dy. \tag{4.4}$$

By using equation (2.9), the term on the right-hand side becomes

$$\frac{v_y}{v} \left(\frac{u_{1y}}{v} \right)_y = \frac{v_y}{v} \left(\frac{v_t}{v} \right)_y$$
$$= \frac{v_y}{v} (log\, v)_{ty}$$
$$= \frac{v_y}{v} \left(\frac{v_y}{v} \right)_t .$$

Equation (4.4) becomes

$$\int_{-\infty}^{\infty} \frac{v_y}{v} \left(\frac{v_t}{v} \right)_y dy - \int_{-\infty}^{\infty} \frac{v_y}{v} u_{1t} dy$$
$$= \int_{-\infty}^{\infty} \frac{v_y}{v} \left(p_\perp (1 - \hat{b_1^2}) + p_{\shortparallel} \hat{b_1^2} \right)_y dy + \int_{-\infty}^{\infty} \frac{v_y}{v} \left(\frac{1}{2}(b_2^2 + b_3^2) \right)_y dy. \tag{4.5}$$

We use the product rule for the second term on the left-hand side of equation (4.5)

$$\frac{d}{dt}(u_1 \frac{v_y}{v}) = u_1 (\frac{v_y}{v})_t + u_{1t} \frac{v_y}{v}.$$

And we use

$$B_y = \frac{b_2 b_{2y} + b_3 b_{3y}}{B}$$

for the derivatives of $\hat{b_2}$ and $\hat{b_3}$. Then integrating equation (4.4) in t, we have

$$\begin{aligned}
&\int_{-\infty}^{\infty} \frac{1}{2}(\frac{v_y}{v})^2(y,t)dy - \int_{-\infty}^{\infty} u_1 \frac{v_y}{v} dy - \int_0^t \int_{-\infty}^{\infty} \frac{u_{1y}^2}{v} dy d\tau \\
&\leq \int_{-\infty}^{\infty} \frac{1}{2}(\frac{v_y}{v})^2(y,0)dy + \int_0^t \int_{-\infty}^{\infty} \frac{2}{v^2 B} b_2 b_{2y} v_y dy d\tau \\
&+ \int_0^t \int_{-\infty}^{\infty} \frac{2}{v^2 B} b_3 b_{3y} v_y dy d\tau - \int_0^t \int_{-\infty}^{\infty} \frac{1}{v^3 B} b_2^2 v_y^2 dy d\tau \\
&- \int_0^t \int_{-\infty}^{\infty} \frac{1}{v^3 B} b_3^2 v_y^2 dy d\tau - \int_0^t \int_{-\infty}^{\infty} \frac{3}{v^5 B^4} v_y^2 dy d\tau \\
&- \int_0^t \int_{-\infty}^{\infty} \frac{1}{v^2 B^3} b_2^2 (b_2 b_{2y} + b_3 b_{3y}) v_y dy d\tau \\
&- \int_0^t \int_{-\infty}^{\infty} \frac{1}{v^2 B^3} b_3^2 (b_2 b_{2y} + b_3 b_{3y}) v_y dy d\tau \\
&- \int_0^t \int_{-\infty}^{\infty} \frac{4}{v^3 B^6} (b_2 b_{2y} + b_3 b_{3y}) v_y dy d\tau \\
&+ \int_0^t \int_{-\infty}^{\infty} \frac{1}{2} \frac{v_y}{v} (b_2^2 + b_3^2)_y dy d\tau.
\end{aligned} \tag{4.6}$$

Now, by adding equations (4.2), (4.3), and (4.6) together, we obtain

$$
\begin{aligned}
&\int_{-\infty}^{\infty}\frac{1}{2}\sum_{1}^{3}u_i^2(y,t)dy+\int_{-\infty}^{\infty}\frac{1}{2}\left(\frac{v_y}{v}\right)^2(y,t)dy+\int_{-\infty}^{\infty}\sum_{i=2}^{3}\frac{1}{2}vb_i^2(y,t)dy\\
&-\int_{-\infty}^{\infty}u_1\frac{v_y}{v}dy+\int_{-\infty}^{\infty}(1-\frac{1}{\bar{v}^2\bar{B}^3})(B-\bar{B})dy+\int_{-\infty}^{\infty}\frac{2}{\bar{v}^4\bar{B}^2}(v-\bar{v})^2dy\\
&+\int_{-\infty}^{\infty}\frac{2}{\bar{v}^2\bar{B}^4}(B-\bar{B})^2dy+\int_0^t\int_{-\infty}^{\infty}\zeta v_y^2dyd\tau+\int_0^t\int_{-\infty}^{\infty}\sum_{i=2}^{3}\frac{u_{iy}^2}{v}dyd\tau\\
&+\int_0^t\int_{-\infty}^{\infty}\sum_{i=2}^{3}\frac{b_{iy}^2}{v}dyd\tau\leq\int_{-\infty}^{\infty}\frac{1}{2}\sum_{1}^{3}u_i^2(y,0)dy+\int_{-\infty}^{\infty}\frac{1}{2}\sum_{i=2}^{3}vb_i^2(y,0)dy\\
&+\int_{-\infty}^{\infty}\frac{1}{2}\left(\frac{v_y}{v}\right)^2(y,0)dy+\int_0^t\int_{-\infty}^{\infty}\sum_{i=2}^{3}\frac{2}{v^2B}b_ib_{iy}v_ydyd\tau\\
&-\int_0^t\int_{-\infty}^{\infty}\sum_{i=2}^{3}\frac{1}{v^2B^3}b_i^2(b_2b_{2y}+b_3b_{3y})v_ydyd\tau\\
&-\int_0^t\int_{-\infty}^{\infty}\frac{4}{v^3B^6}(b_2b_{2y}+b_3b_{3y})v_ydyd\tau\\
&+\int_0^t\int_{-\infty}^{\infty}\frac{1}{2}\frac{v_y^2b_3^2}{v^2}dyd\tau+\int_0^t\int_{-\infty}^{\infty}\frac{1}{2}\frac{v_y}{v}(b_2^2+b_3^2)_ydyd\tau\\
&+\int_0^t\int_{-\infty}^{\infty}\sum_{i=2}^{3}\left(\frac{1}{v}b_{iy}\right)_y\left(-\frac{B}{v}+\frac{1}{v^3B^2}\right)\frac{b_i}{B^2}dyd\tau,
\end{aligned}
\tag{4.7}
$$

where

$$\zeta = \frac{b_2^2}{v^3 B} + \frac{b_3^2}{v^3 B} + \frac{3}{v^5 B^4}.$$

Using Cauchy Schwarz inequality, we estimates of the rest of the terms on the right-hand side of equation (4.7). For example, consider

$$\int_0^t \int_{-\infty}^{\infty} \frac{2}{v^2 B} b_2 b_{2y} v_y dy d\tau$$

$$|\int_0^t \int_{-\infty}^{\infty} \frac{2}{v^2 B} b_2 b_{2y} v_y dy d\tau| \leq C \int_0^t \sup|b_2| \int_{-\infty}^{\infty} |v_y b_{2y}| dy d\tau,$$

by Sobolev embedding theorem, and we have

$$\sup_y |b_2| \leq \left(\int_{-\infty}^{\infty} (b_2^2 + b_{2y}^2) dy \right)^{\frac{1}{2}}$$

Thus,

$$\int_0^t \int_{-\infty}^{\infty} \frac{2}{v^2 B} b_2 b_{2y} v_y dy d\tau$$

$$\leq C \sup_t \left(\int_{-\infty}^{\infty} (b_2^2 + b_{2y}^2) dy \right)^{\frac{1}{2}} \int_0^t \int_{-\infty}^{\infty} (v_y^2 + b_{2y}^2) dy d\tau$$

$$\leq C E(t)^{1/2} F(t).$$

Thus we estimate the terms on the right hand-side of equation (4.7) as $(E(t)^{1/2} + E(t))F(t)$. Therefore,

$$E_1(t) + F_1(t) \leq E_1(0) + C(E(t)^{1/2} + E(t))F(t), \tag{4.8}$$

where

$$E_1(t) = \frac{1}{2} \left\{ \|u\|^2 + \|v - \bar{v}\|^2 + \|q\|^2 + \|\frac{v_y}{v}\|^2 \right\},$$

$$F_1(t) = \int_0^t \left\{ \|u_y\|^2 + \|q_y\|^2 + \|v_y\|^2 \right\} d\tau.$$

Second, we need to estimate the first derivatives of u and q. For that purpose, we multiply equations (2.10), (2.11), and (2.12) by $-u_{1yy}, -u_{2yy}$ and $-u_{3yy}$, respectively. Similarly, we differentiate equations (2.13) and (2.14) in y and multiply them by b_{2y} and b_{3y}, respectively. We combine them and integrate with respect to y and t. We obtain

$$
\begin{aligned}
&\sum_{i=1}^{3}\int_{-\infty}^{\infty}\frac{1}{2}u_{iy}^{2}(y,t)dy+\sum_{i=2}^{3}\int_{-\infty}^{\infty}\frac{1}{2}vb_{iy}^{2}(y,t)dy \\
&+\int_{0}^{t}\int_{-\infty}^{\infty}\sum_{i=1}^{3}\frac{1}{v}u_{iyy}^{2}dyd\tau+\int_{0}^{t}\int_{-\infty}^{\infty}\sum_{i=2}^{3}\frac{b_{iyy}^{2}}{v}dyd\tau \\
&\leq \int_{-\infty}^{\infty}\sum_{i=1}^{3}\frac{1}{2}u_{iy}^{2}(y,0)dy\int_{-\infty}^{\infty}\sum_{i=2}^{3}\frac{vb_{iy}^{2}}{2}(y,0)dy \\
&+\int_{0}^{t}\int_{-\infty}^{\infty}\sum_{i=1}^{3}\frac{v_y}{v^2}u_{1y}u_{iyy}dyd\tau-\int_{0}^{t}\int_{-\infty}^{\infty}\sum_{i=2}^{3}u_{1y}b_{iy}b_{iy}dyd\tau \qquad (4.9)\\
&-\int_{0}^{t}\int_{-\infty}^{\infty}\sum_{i=2}^{3}v_yb_{iy}b_{it}dyd\tau-\int_{0}^{t}\int_{-\infty}^{\infty}\sum_{i=2}^{3}u_{1yy}b_ib_{iy}dyd\tau \\
&+\int_{0}^{t}\int_{-\infty}^{\infty}\sum_{i=2}^{3}\frac{1}{v^2}b_{iy}v_ydyd\tau+2\varepsilon\int_{0}^{t}\int_{-\infty}^{\infty}\sum_{i=1}^{3}u_{iyy}^{2}dyd\tau \\
&+\frac{3C}{\varepsilon}\int_{0}^{t}\int_{-\infty}^{\infty}v_y^2dyd\tau+\frac{3C}{\varepsilon}\int_{0}^{t}\int_{-\infty}^{\infty}B_y^2dyd\tau.
\end{aligned}
$$

For the second derivative of v, take the derivative with respect to y of equation (2.10) and multiply by $\frac{v_{yy}}{v}$. Then we perform the integration with respect to y and t.

$$\int_0^t \int_{-\infty}^{\infty} \frac{v_{yy}}{v} u_{1ty} dy d\tau + \int_0^t \int_{-\infty}^{\infty} \frac{v_{yy}}{v} \left(p_\perp (1 - \hat{b_1^2}) + p_{\shortparallel} \hat{b_1^2} \right)_{yy} dy d\tau$$

$$+ \int_0^t \int_{-\infty}^{\infty} \frac{v_{yy}}{v} \left(\frac{1}{2}(b_2^2 + b_3^2) \right)_{yy} dy d\tau = \int_0^t \int_{-\infty}^{\infty} \frac{v_{yy}}{v} \left(\frac{u_{1y}}{v} \right)_{yy} dy d\tau.$$

However, using equation (2.9), the term on the right-hand side becomes

$$\begin{aligned} \frac{v_{yy}}{v} \left(\frac{u_{1y}}{v} \right)_{yy} &= \frac{v_{yy}}{v} \left(\frac{v_y}{v} \right)_{ty} \\ &= \frac{v_{yy}}{v} (log\ v)_{tyy} \\ &= \frac{v_{yy}}{v} \left(\frac{v_{yy}}{v} - \frac{v_y^2}{v^2} \right)_t . \end{aligned}$$

So,

$$\int_0^t \int_{-\infty}^{\infty} \frac{v_{yy}}{v} \left(\frac{v_{yy}}{v} \right)_t dy d\tau - \int_0^t \int_{-\infty}^{\infty} \frac{v_{yy}}{v} \left(\frac{v_y^2}{v^2} \right)_t dy d\tau - \int_0^t \int_{-\infty}^{\infty} \frac{v_{yy} v_{tt}}{v} dy d\tau$$

$$= \int_0^t \int_{-\infty}^{\infty} \frac{v_{yy}}{v} \left(p_\perp (1 - \hat{b_1^2}) + p_{\shortparallel} \hat{b_1^2} \right)_{yy} dy d\tau$$

$$+ \int_0^t \int_{-\infty}^{\infty} \frac{v_{yy}}{v} \left(\frac{1}{2}(b_2^2 + b_3^2) \right)_{yy} dy d\tau.$$

For the estimate of v_{yy}, we perform a few estimates. The third term on the left-hand side of the above equation becomes

$$\int_0^t\int_{-\infty}^{\infty}\frac{v_{yy}v_{tt}}{v}dyd\tau=\int_{-\infty}^{\infty}\frac{v_{yy}}{v}v_t dy-\int_0^t\int_{-\infty}^{\infty}\left(\frac{v_{yy}}{v}\right)_t v_t dyd\tau$$

$$=\int_{-\infty}^{\infty}\frac{v_{yy}}{v}v_t dy-\int_0^t\int_{-\infty}^{\infty}\frac{v_{yyt}}{v}v_t dyd\tau+\int_0^t\int_{-\infty}^{\infty}\frac{v_{yy}v_t^2}{v^2}dyd\tau.$$

We estimate the third term on the right-hand side from the above equation, and by using equation (2.9), we have

$$\int_0^t\int_{-\infty}^{\infty}\frac{v_{yy}v_t^2}{v^2}dyd\tau=\int_0^t\int_{-\infty}^{\infty}\left(\frac{2v_y u_{1y}u_{1yy}}{v^2}-\frac{2v_y^2u_{1y}^2}{v^3}\right)dyd\tau$$

Therefore,

$$\int_{-\infty}^{\infty}\frac{1}{2}\left(\frac{v_{yy}}{v}\right)^2(y,t)dy-\int_{-\infty}^{\infty}\frac{v_{yy}}{v}v_t dy$$

$$\leq\int_{-\infty}^{\infty}\frac{1}{2}\left(\frac{v_{yy}}{v}\right)^2(y,0)dy+\int_0^t\int_{-\infty}^{\infty}\frac{v_{yy}}{v}\left(\frac{v_y^2}{v^2}\right)_t dyd\tau$$

$$-\int_0^t\int_{-\infty}^{\infty}\frac{v_{yyt}}{v}v_t dyd\tau+\int_0^t\int_{-\infty}^{\infty}\frac{2v_y u_{1y}u_{1yy}}{v^2}dyd\tau$$

$$-\int_0^t\int_{-\infty}^{\infty}\frac{2v_y^2u_{1y}^2}{v^3}dyd\tau+\int_0^t\int_{-\infty}^{\infty}\left(\frac{v_{yy}}{v}\right)\left(\frac{B}{v}(1-\frac{1}{B^2})+\frac{1}{v^3B^4}\right)_{yy}dyd\tau$$

$$+\int_0^t\int_{-\infty}^{\infty}(\frac{v_{yy}}{v})\left(\frac{1}{2}(b_2^2+b_3^2)\right)_{yy}dyd\tau.$$

Take the second derivative with respect to y of the last two terms on the right-hand side of the above equation. Then, we have

$$
\begin{aligned}
&\int_{-\infty}^{\infty}\frac{1}{2}\left(\frac{v_{yy}}{v}\right)^2(y,t)dy-\int_{-\infty}^{\infty}\frac{v_{yy}}{v}v_t dy+\int_0^t\int_{-\infty}^{\infty}v_{yy}^2\left[\frac{3}{v^5B^4}+\frac{B}{v^3}-\frac{1}{v^3B}\right]dyd\tau\\
&\le\int_{-\infty}^{\infty}\frac{1}{2}\left(\frac{v_{yy}}{v}\right)^2(y,0)dy+\int_0^t\int_{-\infty}^{\infty}\frac{2v_yu_{1yy}v_{yy}}{v^3}dyd\tau\\
&-\int_0^t\int_{-\infty}^{\infty}\frac{2u_{1y}v_y^2v_{yy}}{v^4}dyd\tau-\int_0^t\int_{-\infty}^{\infty}\frac{3u_{1y}u_{1yy}v_y}{v^2}dyd\tau\\
&-\int_0^t\int_{-\infty}^{\infty}\frac{2v_y^2u_{1y}^2}{v^3}dyd\tau+\int_0^t\int_{-\infty}^{\infty}\frac{1}{v^2}v_{yy}B_{yy}dyd\tau\\
&-\int_0^t\int_{-\infty}^{\infty}\frac{2}{v^3}v_yv_{yy}B_ydyd\tau+\int_0^t\int_{-\infty}^{\infty}\frac{2}{v^4}v_y^2v_{yy}Bdyd\tau\\
&+\int_0^t\int_{-\infty}^{\infty}\frac{1}{v^2B^2}v_{yy}B_{yy}dyd\tau-\int_0^t\int_{-\infty}^{\infty}\frac{2}{v^2B^3}v_{yy}{B_y}^2dyd\tau\\
&-\int_0^t\int_{-\infty}^{\infty}\frac{2}{v^3B^2}v_yv_{yy}B_ydyd\tau-\int_0^t\int_{-\infty}^{\infty}\frac{2}{v^4B}{v_y}^2v_{yy}dyd\tau\\
&-\int_0^t\int_{-\infty}^{\infty}\frac{4}{v^4B^5}v_{yy}B_{yy}dyd\tau-\int_0^t\int_{-\infty}^{\infty}\frac{20}{v^4B^6}v_{yy}{B_y}^2dyd\tau\\
&+\int_0^t\int_{-\infty}^{\infty}\frac{24}{v^5B^5}v_yv_{yy}B_ydyd\tau+\int_0^t\int_{-\infty}^{\infty}\frac{12}{v^6B^4}{v_y}^2v_{yy}dyd\tau\\
&+\int_0^t\int_{-\infty}^{\infty}\sum_{i=2}^{3}\frac{v_{yy}}{v}b_ib_{iyy}dyd\tau+\int_0^t\int_{-\infty}^{\infty}\sum_{i=2}^{3}\frac{v_{yy}}{v}b_{iy}b_{iy}dyd\tau.
\end{aligned}
\tag{4.10}
$$

We combine equations (4.9) and (4.10) and we estimate the terms on the right-hand side as $(E(t)^{1/2}+E(t))F(t)$. Therefore,

$$E_2(t)+F_2(t) \leq E_2(0)+C(E(t)^{1/2}+E(t))F(t), \tag{4.11}$$

where

$$E_2(t)=\frac{1}{2}\left\{\|u_y\|^2+\|q_y\|^2+\|\frac{v_{yy}}{v}\|^2\right\},$$

$$F_2(t)=\int_0^t\left\{\|u_{yy}\|^2+\|q_{yy}\|^2+\|v_{yy}\|^2\right\}d\tau.$$

Third, we differentiate equations (2.10), (2.11) and (2.12) in y, multiply them by $-u_{1yyy}, -u_{2yyy}$ and $-u_{3yyy}$, similarly, we differentiate equations (2.13) and (2.14) in y, and multiply by $-b_{2yyy}$ and $-b_{3yyy}$, respectively, and integrate with respect to y and t. Then, we combine them. We obtain

$$
\begin{aligned}
&\int_{-\infty}^{\infty}\frac{1}{2}\sum_{1}^{3}u_{iyy}^2(y,t)dy+\sum_{i=2}^{3}\int_{-\infty}^{\infty}\frac{1}{2}vb_{iyy}^2(y,t)dy\\
&+\int_0^t\int_{-\infty}^{\infty}\sum_{1}^{3}\frac{u_{iyyy}^2}{v}dyd\tau+\int_0^t\int_{-\infty}^{\infty}\sum_{i=2}^{3}\frac{b_{iyyy}^2}{v}dyd\tau\\
&\leq\int_{-\infty}^{\infty}\frac{1}{2}\sum_{1}^{3}u_{iyy}^2(y,0)dy+\int_{-\infty}^{\infty}\sum_{i=2}^{3}\frac{1}{2}vb_{iyy}^2(y,0)dy\\
&+\int_0^t\int_{-\infty}^{\infty}\sum_{1}^{3}\frac{2}{v^2}u_{iyy}u_{iyyy}v_ydyd\tau-\int_0^t\int_{-\infty}^{\infty}\sum_{1}^{3}\frac{v_{yy}}{v^2}u_{iy}u_{iyyy}dyd\tau\\
&+\int_0^t\int_{-\infty}^{\infty}\sum_{1}^{3}\frac{2v_y}{v^3}u_{iy}u_{iyyy}dyd\tau+\int_0^t\int_{-\infty}^{\infty}\sum_{i=2}^{3}\frac{2}{v^2}v_yb_{iyy}b_{iyyy}dyd\tau\\
&+\int_0^t\int_{-\infty}^{\infty}\sum_{i=2}^{3}\frac{1}{v^2}v_{yy}b_{iy}b_{iyyy}dyd\tau+\int_0^t\int_{-\infty}^{\infty}\frac{2}{v^3}\sum_{i=2}^{3}v_y^2b_{iy}b_{iyyy}dyd\tau\\
&+\int_0^t\int_{-\infty}^{\infty}\sum_{i=2}^{3}v_yb_{iyyy}b_{it}dyd\tau+\int_0^t\int_{-\infty}^{\infty}\sum_{i=2}^{3}u_{1y}b_{iyyy}b_{iy}dyd\tau\\
&+\int_0^t\int_{-\infty}^{\infty}\sum_{i=2}^{3}u_{1yy}b_ib_{iyyy}dyd\tau+\int_0^t\int_{-\infty}^{\infty}u_{1yyy}\left(\frac{1}{2}(b_2^2+b_3^2)\right)_{yy}dyd\tau\\
&+\int_0^t\int_{-\infty}^{\infty}u_{1yyy}\left(-\frac{B}{v}(1-\frac{b_1}{B})+\frac{1}{v^3B^2}\frac{b_1}{B}\right)_{yy}dyd\tau\\
&+\int_0^t\int_{-\infty}^{\infty}\sum_{i=2}^{3}u_{iyyy}\left((-\frac{B}{v}+\frac{1}{v^3B^2})\frac{b_i}{B}\right)_{yy}dyd\tau\\
&-\int_0^t\int_{-\infty}^{\infty}u_{2yyy}\left(\frac{1}{\Omega}(\frac{B}{v}-\frac{2}{v^3B^2})\frac{u_{3y}}{v}\right)_{yy}dyd\tau\\
&+\int_0^t\int_{-\infty}^{\infty}u_{3yyy}\left(\frac{1}{\Omega}(\frac{B}{v}-\frac{2}{v^3B^2})\frac{u_{3y}}{v}\right)_{yy}dyd\tau.
\end{aligned}
\tag{4.12}
$$

We estimate some of the terms on the right-hand side of equation (4.12)

by the Cauchy–Schwartz inequality as $(E(t)^{1/2}+E(t))F(t)$.
Therefore,

$$E_3(t)+F_3(t)\leq E_3(0)+C(E(t)^{1/2}+E(t))F(t). \tag{4.13}$$

where

$$E_3(t)=\frac{1}{2}\left\{\|u_{yy}\|^2+\|q_{yy}\|^2\right\},$$

$$F_3(t)=\int_0^t\left\{\|u_{yyy}\|^2+\|q_{yyy}\|^2\right\}d\tau.$$

Finally, by adding equations (4.8), (4.11) and (4.13). We obtain the following a priori estimate

$$\|(u,v-\bar{v},q)(t)\|_2^2+\int_0^t\left[\sum_{k=1}^{2}\|D^k(u,v,q)(t)\|^2+\|D^3(u,q)(t)\|^2\right]d\tau$$

$$\leqslant C\|(u,v-\bar{v},q)(0)\|_2^2+C(E(t)^{1/2}+E(t))F(t), \tag{4.14}$$

where C is a constant independent of t.

Chapter 5

Global Existence and Asympotatic Behavior

In this chapter, we investigate the global existence of solution for non-linear MHD equations for a give initial data. We state and prove our global existence result of the system (2.9)-(2.14) by combining local solution and a priori estimate and also show the asymptotic behavior of the solution. We state the main result is in the following Theorem.

Theorem 1. *Suppose the initial data $(u, v-\bar{v}, q)(0) \in H^2$. Then, the initial value problem for equations (2.9)-(2.14) has a unique solution $(u, v-\bar{v}, q)(t)$ globally in time such that $(u, v-\bar{v}, q)(t) \in C^0(0, \infty; H^2)$, $D(u, q) \in L^2(0, \infty; H^2)$, $D(v) \in L^2(0, \infty; H^1)$ for $t > 0$ and has estimate for any $t \geq 0$.*

$$\|(u, v-\bar{v}, q)(t)\|_2^2 + \int_0^t \left[\sum_{k=1}^{2} \|D^k(u, v, q)(\tau)\|^2 + \|D^3(u, q)(\tau)\|^2\right] d\tau$$

$$\leqslant C\|(u, v-\bar{v}, q)(0)\|_2^2. \tag{5.1}$$

Furthermore, the solution has the following decay property

$$\|(u_y, v_y, q_y)(t)\| \to 0,$$

$$\|(u, v-\bar{v}, q)(t)\|_{L^\infty} \to 0 \;\; as \;\; t \to \infty. \tag{5.2}$$

Proof. To complete the proof of existence of global solutions, we use induction on the local solution and extends the time, we also prove that solution exists for all time and satisfies (5.1).
If M is small and if initial data $t = 0$ satisfies (5.1), then there exists a local solution and therefore there exists $T_1 > 0$ such that the solution exists and satisfies

$$\sup_{0 \leq t \leq T_1} \|(u, v-\bar{v}, q)(t)\|_2 < M. \tag{5.3}$$

Then, we see that the local solution exists for $t \leq 2T_1$ and satisfies a priori estimate (5.1). Therefore, the Cauchy problem has a solution

$$(u, v, q) \in (T_1, 2T_1; H^2),$$

satisfying the estimate (5.3). Then, the energy estimate for the local solution and the a priori estimate, imply that for $T_1 \leq t \leq 2T_1$ with the initial data $(u, v-\bar{v}, q)(T_1)$, the Cauchy problem (2.9)-(2.15) has a local solution satisfying

$$\sup_{T_1 \leq t \leq 2T_1} \|(u, v-\bar{v}, q)(t)\|_2 \leq \|(u, v-\bar{v}, q)(T_1)\|_2 < M.$$

Then, a priori estimates hold for all t. We continue with the same process for $0 \leqslant t \leqslant nT, \quad n = 3, 4, \ldots$ Thus, we have a global solution $(u, v, q)(t) \in C(0, \infty; H^2)$ which satifies the estimate for all $t \geqslant 0$. This completes the induction and proves that a solution (u, v, q) exists for all time and satisfies the estimate (5.1).
To prove the assertion (5.2) concerning the large-time behavior we show the decay estimates in details for u.

Set

$$f(t) = \int_{-\infty}^{\infty} u_y^2(y,t)dy.$$

Then, since f is a function of bounded variation, by the Cauchy Schwarz inequality, we have

$$\begin{aligned}
\underset{0<s<t}{TV} f(t) &\leq |\int_0^t f'(s)ds| \leq 2\int_0^t \int_{-\infty}^{\infty} |u_y u_{ys}|dyds \\
&\leq 2\int_0^t \|u_y\|\|u_{ys}\|ds \\
&\leq \sup_{0\leq t\leq T} \int_0^t \int_{-\infty}^{\infty} (u_y^2 + u_{ys}^2)dyds
\end{aligned}$$

This implies $f(t)$ has a limit as $t \to \infty$ and it must be zero. Similar conclusion holds for $v - \bar{v}$ and q. Therefore,

$$\|(u_y, v_y, q_y)(t)\| \to 0.$$

For the decay estimate of L^∞ norms, using the following inequality

$$\|u\|_{L^\infty}^2 = \sup_y u^2 \leq 2\int_{-\infty}^{\infty} |uu_y|dy \leq 2\|u\|\|u_y\|,$$

we have

$$\|(u, v - \bar{v}, q)(t)\|_{L^\infty} \to 0.$$

■

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